Geometry of Locally Finite Spaces

Presentation of a new monograph

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Abstract.

The book presents an axiomatic approach to the topology and geometry of locally finite spaces with applications to image processing, computer graphics and to other research areas. Special emphasis is placed on computer solutions of topological and geometrical problems. Basic topological notions such as connectedness, frontier, opening frontier, topological ball, sphere, manifold with and without boundary etc. are defined for locally finite spaces independently of the topology of Euclidean space. The author depicts ways for an explicit computer representation of topological spaces whose properties correspond to the axioms of classical topology. He introduces a new concept of digital geometry based on the topology of locally finite spaces. The concept is independent from Euclidean geometry. The author also introduces a new notion of continuous connectedness preserving maps, which substitute continuous functions in topology and geometry of locally finite spaces. New data structures and numerous geometric and topological algorithms are presented. Most algorithms are accompanied by a pseudo-code based on the C++ programming language.

Locally Finite Spaces

Locally finite spaces serve to overcome the existing discrepancy between theory and applications: The traditional way of research consists in making theory in Euclidean space while applications deal only with finite discrete sets. The reason is that even a small subset of a Euclidean space cannon be explicitly represented in a computer because such a subset, no matter how small it is, contains infinitely many points.

Locally finite spaces are on one hand theoretically consistent and conform with classical topology and on the other hand explicitly representable in a computer.

Aims of the Monograph

The author wishes to demonstrate that it is possible to develop a locally finite topology well suited for applications in computer imagery and independent of the topology of the Euclidean space.

The second aim is to present some advises for developing efficient algorithms in computer imagery based on the topology and geometry of locally finite spaces, in particular of abstract cell complexes. Numerous algorithms of that kind are presented in the monograph.

The main topics of the monograph are:

- Axiomatic Approach to Digital Topology;
- Abstract Cell Complexes – an Important Particular Case;
- Continuous Mappings among Locally Finite Spaces;
- Digital Lines and Planes;
- Theory of Surfaces in a Three-Dimensional Space;
- Data Structures;
- A Universal Algorithm for Tracing Boundaries in *n*D spaces;
- Labeling Connected Components;
- Tracing, Encoding and Reconstructing Surfaces in 3D spaces;
- Topics for Discussion – Irrational Numbers; Optimal Estimates of Derivatives;
- Problems to Be Solved.

New Axioms

Why was a new set of axioms suggested?

The relation of axioms of the classical topology to the demands of computer imagery is not clear for a non-topologist. It is e.g. not clear, why do we need the notion of open subsets satisfying classical axioms.

The new axioms are related to the notions of connectedness and to that of the boundary of a subset. These notions are important for applications, in particular for image analysis.

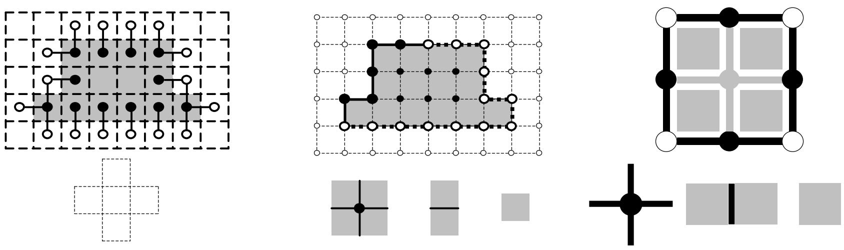
We have demonstrated, that classical axioms can be deduced from the new axioms as theorems. In this way classical axioms become related to the desired properties of connectedness and of boundaries.

Axiom 1: For each space element *e* of the space *S* there are certain subsets containing *e*, which are neighborhoods of *e*. The intersection of two neighborhoods of *e* is again a neighborhood of *e*. Each element *e* has its smallest neighborhood SN(*e*).

Axiom 2: There are space elements, which have in their SN more than one element.

Axiom 3: The frontier Fr(*T*, *S*) of any subset $T \subset S$ is thin.

The notion of a thin frontier is exactly defined in the book. Fig. 1a and 1b illustrate this notion. In Fig. 1a space elements are squares with the well-known 4-neighborhood relation. The frontier of the shaded area consists of the squares labeled by black and white disks. The frontier is labeled by black and white disks; it is thick. In Fig. 1b space elements are squares, lines and dots. The frontier of the shaded area consists of bold lines, both solid and dotted, and of dots labeled by black and white disks. It is thin.



a b c

Fig. 1 Examples of frontiers:
A thick frontier (a); a thin frontier (b); a frontier with gaps (c)

Axiom 4: The frontier of Fr(*T*, *S*) is the same as Fr(*T*, *S*), i.e. Fr(Fr(*T*, *S*), *S*)=Fr(*T*, *S*).

Fig. 1c illustrates the case not satisfying Axiom 4. An important property of the frontier is, non-rigorously speaking, that it must have no gaps, which is not the same, as to say that it must be connected. More precisely, this means that the frontier of a frontier *F* is the same as *F*. For example, the frontier in Fig. 1c has gaps represented by white disks. Let us explain this. Fig. 1c shows a space *S* consisting of squares, lines and dots. The neighborhood relation *N* is in this case non-transitive: The neighborhood SN(*P*) of a dot *P* contains some lines incident to *P* but no squares. The SN of a line contains one or two incident squares, while the SN of a square is the square itself. The subset *T* under consideration is represented by gray elements. Its frontier Fr(*T*, *S*) consists of black lines and black dots (disks) since these elements do not belong to *T*, while their SNs intersect *T*. The white dots do not belong to *F*=Fr(*T*, *S*) because their SNs do not intersect *T*. These are the gaps. However, Fr(*F*, *S*) contains the white dots because their SNs intersect both F and its complement (at the dots themselves). Thus in this case the frontier F=Fr(T, S) is different from Fr(F, S).

Properties of ALF Spaces

We call a locally finite space satisfying our Axioms an ALF space. We have demonstrated in Section 2.3 of the book that the classical axioms can be deduced as theorems from our Axioms and that an ALF space is a *particular case* of the classical T_0 space, but not of a T_1 space.

An abstract cell complex is a particular case of an ALF space characterized by an additional feature: the dimension function *dim*(a), which assigns a non-negative integer to each space element in such a way that if $b \in$ SN(a), then $dim(a) \leq dim(b)$. Elements of an AC complex are called *cells*. We use the well-known definition of abstract cell complexes suggested by Steinitz [Stein08]:

Definition AC: An *abstract cell complex* (AC complex) $C=(E, B, dim)$ is a set E of abstract elements (cells) provided with an asymmetric, irreflexive, and transitive binary relation $B \subset E \times E$ called the *bounding relation*, and with a dimension function $dim: E \rightarrow I$ from E into the set I of non-negative integers such that $dim(e') < dim(e'')$ for all pairs $(e', e'') \in B$.

A cell is never a subset of another cell.

We have augmented the above definition by a topological definition of the dimensions of space elements. Dimensions of cells represent the partial order corresponding to the bounding relation. Let us call the sequence $a<b< \ldots <k$ of cells of a complex C, in which each cell bounds the next one, a *bounding path* from a to k in C. The number of cells in the sequence *minus one* is called the *length* of the bounding path.

Definition DC (dimension of a cell): The dimension $dim(c, C)$ of a cell c of a complex C is the length of the *longest bounding path* from any element of C to c.

This definition is in correspondence with the well-known notion of the topological dimension or height of an element of a partially ordered set [Birk61].

According to Definition DC the dimension of a cell c is defined *relative* to a subcomplex containing the cell c because the length of the longest bounding path can be different in different subcomplexes.

An example of calculating the dimensions of cells is shown in Fig.2. The cell v has dimension 3 since the length of the path $p<e<f<v$ is equal to 3.

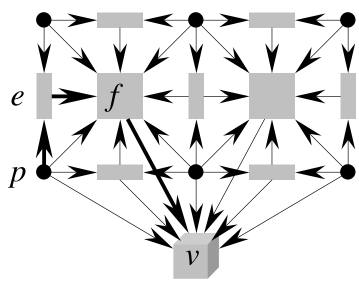


Fig. 2 A complex with bounding relations represented by arrows.
An arrow points from a to b if a bounds b

The dimension of the space elements is an important property. Using dimensions prevents one from errors which can occur when using an LF space without dimensions. An example of a typical error is presented in the book.

We have introduced the notion of an n-dimensional Cartesian complex as the Cartesian product of n one-dimensional complexes [Kov86]. This gives us the possibility to define coordinates of the cells. We call them *combinatorial coordinates*. Fig. 3 shows the closures and the smallest neighborhoods (SONs) of cells of Cartesian complexes of different dimensions.

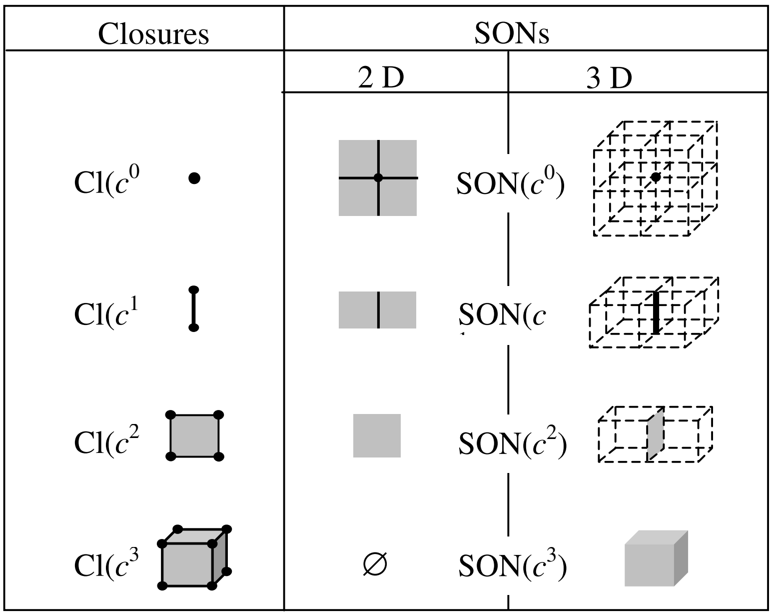


Fig. 3 Closures and SONs of cells of Cartesian AC complexes

Combinatorial Homeomorphism, Balls and Spheres

The notion of the homeomorphism of two sets is a fundamental notion of topology: Two sets are called homeomorphic or topologically equivalent if there is a continuous bijection from one to the other whose inverse is also continuous. There is another classical way to define the homeomorphism. It is directly applicable to complexes and can be extended to other locally finite spaces. It is called the *combinatorial homeomorphism* and is based on the notion of *elementary subdivisions* [Stil95, p. 24]. We shall apply it to AC complexes.

The original concept of an AC complex is too general: It is e.g. possible to define a "strange" AC complex with a 1-cell bounded by more than two 0-cells, or with a 2-cell that has a hole, or with a 3-cell being a torus, etc. All this does not contradict the above Definition AC. To avoid such situations elementary subdivisions have been defined in classical topology (see e.g. a modern survey in [Stil95]) on the base of the topology of Euclidean space and of Euclidean complexes. Since our aim is to develop a theory independent of the theory of Euclidean spaces, we shall suggest new definitions based exclusively on the topology of AC complexes. We suppose that the notion of the combinatorial homeomorphism is not applicable to any complex. There must be a limitation excluding "strange" complexes as mentioned above. This limitation should be of the same nature as the classical limitation defining Euclidean cells as convex sets.

One possible way is to try to introduce a class of complexes which are in certain sense similar to Cartesian ones since Cartesian complexes have the desired properties: A 1-cell is bounded by no more than two 0-cells; a 2-cell has no holes; a 3-cell is a topological three-dimensional ball, etc. However, we do not see a possibility to define the class of complexes homeomorphic to Cartesian ones before having defined the notions of a topological ball and a topological sphere, which are necessary to define the subdivisions. Our intention is to define topological notions *before* geometric ones, because we believe that geometry can be consistently constructed only after the corresponding topological space is already defined. Topology must be the foundation of geometry and not vice versa. Therefore we do not employ geometric notions like metric and Euclidean coordinates in topological definitions. Thus we cannot employ the classical definition of a topological ball, which is a set of points having a limited *distance* to a center point.

We have accommodated the notions of a topological ball and of a topological sphere to complexes, while introducing the notions of *combinatorial balls and spheres* which we denote by AC balls and spheres correspondingly. To avoid the usage of "strange" complexes we have introduced the notions of *proper* cells and complexes which can be regarded as a substitute for convex cells of Euclidean complexes.

The notion of a proper cell has lead to new definitions of balls and spheres independent of geometry and metric. The following Fig. 4 is an illustration to the notions of combinatorial balls and spheres.

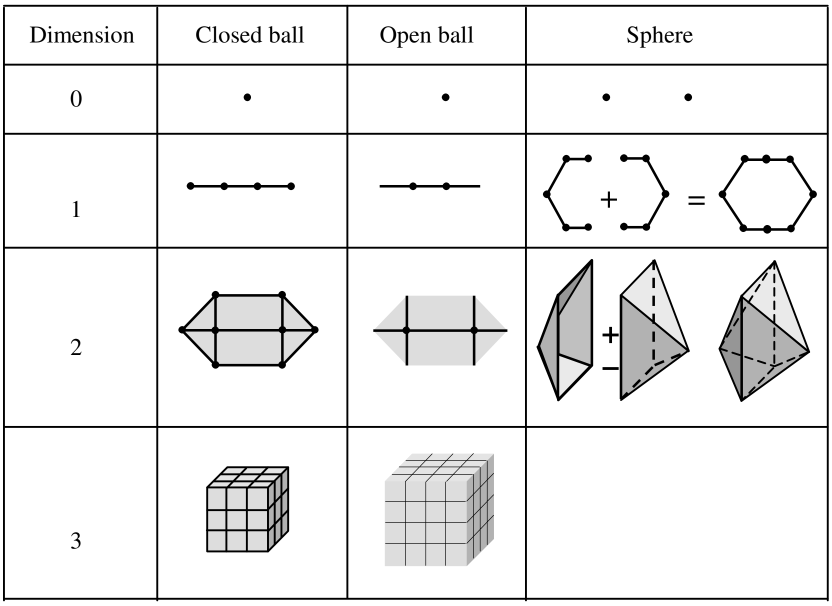


Fig. 4 Examples of AC balls and spheres of dimensions from 0 to 3

The purely combinatorial definitions of a topological ball and a topological sphere independently of the Euclidean space have provided the possibility to justify the well-known notion of combinatorial homeomorphism for locally finite spaces. It is based on elementary subdivisions. Fig. 5 shows an example of the elementary subdivision of a two-dimensional cell.

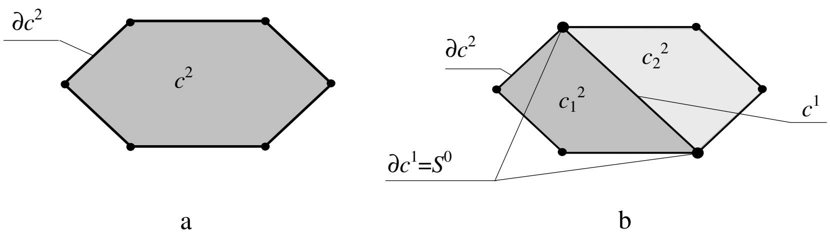


Fig. 5 An example of the elementary subdivision of a 2-cell; the original cell (a) and its subdivision (b)

Exact definitions and proofs are to be found in the book.

Fig. 6 presents an example demonstrating the combinatorial homeomorphism of a square and a triangle.

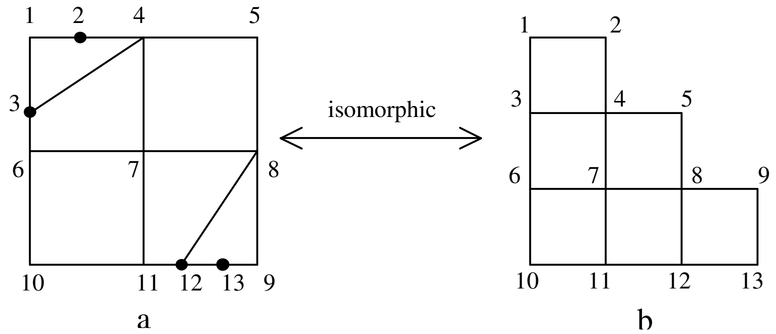


Fig. 6 A subdivision of a digitized square (a) which is isomorphic to a digitized triangle (b); "fat points" denote the new points introduced during the subdivision

We also have generalized the notions of a boundary and a frontier while having introduced the notions of an *opening frontier, opening boundary and generalized boundary*. Thus for example a punctured two-dimensional sphere (i.e. a sphere without a point) is from the point of view of the new definitions a manifold with an opening boundary while from the classical point of view it is no manifold at all.

Continuous Functions and Connectivity Preserving Maps

In classical topology homeomorphism is defined by means of continuous mappings between topological spaces. The possibilities to apply this idea to locally finite spaces are rather limited: We have demonstrated in Section 4.2 that isomorphism is the only *classical* homeomorphism possible between two locally finite spaces (LFS)

We have demonstrated that it is impossible to continuously map one LFS onto a "greater" space, i.e. onto a space containing more elements. We have suggested to consider more general correspondences between *X* and *Y*, assigning to each cell of *X* a *subset of Y* rather than a single cell [Kov93, Kov94]. A connectivity preserving map is continuous if the preimage of each open subset is open. Fig. 7 shows two examples of CPMs.

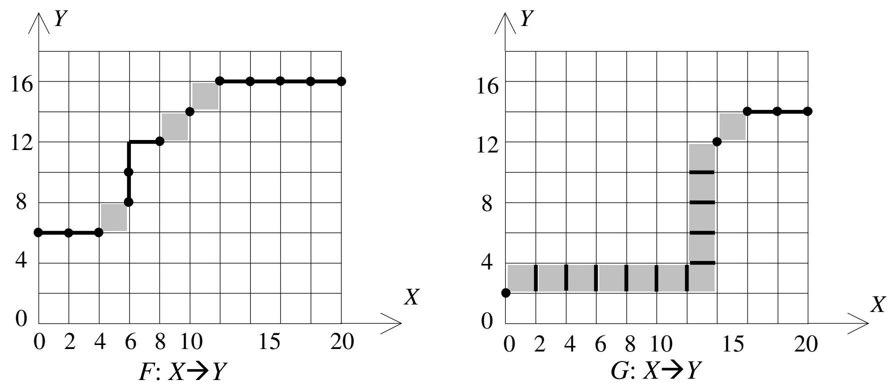


Fig. 7 Examples of correspondences: *F* is connectedness preserving, simple and not continuous; *G* is continuous and not simple.

We have also demonstrated that combinatorial homeomorphism *X*~*Y* according to the Definition CH (Section 3.6, p. 57) uniquely specifies a continuous CPM *F*: *X*→*Y* whose inverse correspondence is also a continuous CPM.

Digital Geometry. Digital Lines and Planes

Sections 6 and 7 of the book contain definition of digital lines and planes. The definitions are independent of the corresponding Euclidean notions. This means that a digital line is defined not as the result of digitizing a Euclidean line. We define a half-plane as a set of elements whose combinatorial coordinates satisfy a linear inequality and a digital straight segment as a connected subset of the boundary of a half-plane. We distinguish between two types of digital curves in a two-dimensional space: visual curves are sequences of pixels (two-dimensional cells) and are well suited for representing curves in an image; boundary curves are sequences of zero- and one-dimensional cells and are well suited for purposes of image analysis. We consider mainly boundary lines rather than visual lines.

Section 7 of the book presents a complete theory of digital straight segments (DSS) being regarded as boundary lines. Equation defining such a DSS and the algorithm of recognizing a DSS are similar to the well-known equations and algorithms for visual lines, but there are also some important differences.

A fast algorithm for subdividing a digital boundary curve into longest DSS is presented in this section. A method of economically and loss-free encoding sequences of DSSs is described in Section 7.2.5.

Section 7.3 contains the theory of digital planes; Section 8 – that of surfaces in a three-dimensional space and Section 9 – the theory of digital arcs.

Applications of the DSSs

1. Estimating the perimeter of a subset.
2. Representing objects in 2D images as polygons for the purpose of shape analysis.
3. Economical and exact encoding of images.There exist many different DSSs going through given two points. To distinguish them three additional integer parameters *L, M, N* must be specified. *M/N* is the slope of the base of the DSS, *L* is the value of the left side $H(x, y)$ of the equation $H(x, y)=0$ of the base at the starting point. A DSS can be *exactly* reconstructed from its end points and the additional parameters. There is a possibility to economically encode a sequence of DSSs while using at the average 2.3 byte per DSS. Fig. 8 presents an example of fast encoding of an image by DSS polygons and of recognizing all disk-shaped objects.

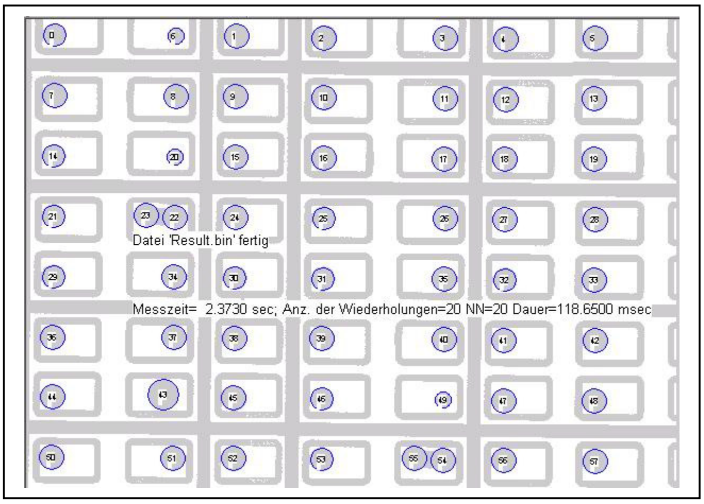


Fig. 8 Example of an image of a wafer with recognized disk-shaped objects

The binary image of 832×654 pixels shown in Fig. 8 was encoded by DSSs. Distorted disk-shaped objects in the image were recognized. The whole processing of encoding the image and recognizing 57 objects took 20 ms on a PC with a Pentium processor of 700 MHz

Applications and Algorithms

Section 11 of the book starts with *recommendations for designers of algorithms*. We recommend not to use adjacency relations, but rather to consider all topological and geometrical problems from the point of view of locally finite topological spaces (ALF spaces) or, even better, of AC complexes. Complex have some advantages as compared to other ALF spaces due to the presence of the dimensions of cells. The dimensions of cells make the work with a topological space easier and more illustrative and help avoiding contradictions. Section 11.1 contains concrete recommendations for ways of using AC complexes for the development of algorithms in computer imagery.

Section 11.2 contains descriptions of various algorithms for tracing and encoding boundaries in 2D images and in 2D subspaces of n-dimensional spaces. Among them is the universal algorithm for tracing boundaries of 2D slices in nD images; n=2, 3, 4; and the algorithm for generating the *block cell list* of a segmented multicolored image. The block cell list is a data structure developed by the author [Kov89]. It enables an economical and loss-free encoding of the image and is well suited for the image analysis because it contains the full topological and geometrical information. Relations between subsets of the image are available without a search.

Sections 11.3 to 11.7 of the book contain descriptions of the following algorithms:

1) Loss-free encoding of digital straight segment with additional parameters;

2) Exact reconstruction of n-dimensional images from boundary codes; n=2, 3, 4;

3) Labeling connected components, two different algorithms;

4) Computing skeletons of subsets in 2D and 3D;

5) Algorithm for topological investigations.

Section 12 describes the method of constructing convex hulls in three dimensional spaces.

Section 13 is devoted to tracing and encoding of surfaces in a three dimensional space. It contains descriptions of the following four algorithms:

a) The simplest encoding of a surface by the depth-first search;

b) Loss-les encoding of a surface by its Euler circuit;

c) Spiral tracing method producing a handle decomposition of the surface along with its code;

d) Economical encoding of surfaces by the "Hoop Code" with less than 2 bits per facet.

Most algorithms are accompanied by a pseudo-code based on the C++ programming language.

Topics for Discussions and Problems to be Solved

The last section of the monograph is devoted to disputable questions of the necessity and possibility to avoid the usage of irrational numbers and of the optimal method of calculating derivatives of functions, that are defined with a limited precision.

Most sections of the monograph present some problems to be solved.

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